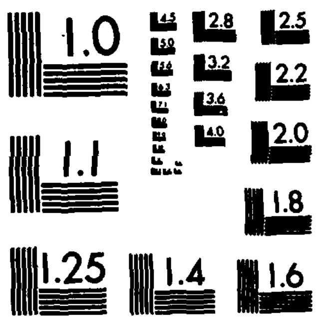

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The composite control proposed in an earlier paper for a class of singularly perturbed nonlinear systems is now shown to possess properties essential for near-optimal feedback design. It asymptotically stabilizes the desired equilibrium and produces a finite cost which tends to the optimal cost for a slow problem as the singular perturbation parameter tends to zero. Thus the well-posedness of the full regulator problem is established. The stability results are also applicable to two-time scale systems which are not singularly perturbed, and the paper does not assume the knowledge of singular perturbation

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20. Abstract (continued)

cont techniques.

Composite control originally proposed in a deterministic context is generalized to the problem with white noise inputs. However, the approach used here is radically different from the deterministic approach. Presence of noise smoothed the system behavior and allowed a more complete solution than in the deterministic case.

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STABILIZATION AND STOCHASTIC CONTROL OF A CLASS OF NONLINEAR SYSTEMS

by

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J. H. Chow
P. V. Kokotovic

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Preface

This report consists of two papers dealing with optimal control of systems with slow nonlinearities modeled as singularly perturbed systems. In the method developed a composite control is designed in two stages. A slow nonlinear and a fast linear subproblem need to be solved.

The first paper by Chow and Kokotovic establishes stabilizing and near optimality properties of the composite control in the deterministic case. In the second paper by Bensoussan the same system is considered with white noise disturbance inputs. The presence of noise smoothed the system behavior and allowed a more complete solution than in the deterministic case.

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A TWO STAGE LYAPUNOV-BELLMAN FEEDBACK DESIGN
OF A CLASS OF NONLINEAR SYSTEMS*

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ABSTRACT

The composite control proposed in an earlier paper for a class of singularly perturbed nonlinear systems is now shown to possess properties essential for near-optimal feedback design. It asymptotically stabilizes the desired equilibrium and produces a finite cost which tends to the optimal cost for a slow problem as the singular perturbation parameter tends to zero. Thus the well-posedness of the full regulator problem is established. The stability results are also applicable to two-time scale systems which are not singularly perturbed, and the paper does not assume the knowledge of singular perturbation techniques.

*The work of P. V. Kokotovic was supported in part by the U. S. Air Force under Grant AFOSR-78-3633, in part by the Joint Services Electronics Program (U. S. Army, U. S. Navy, and U. S. Air Force) under Contract N00014-79-C-0424, and in part by the National Science Foundation under Grant ECS-79-19396. Part of this work was performed when J. H. Chow was a Research Associate at the Coordinated Science Laboratory, University of Illinois.

1. Introduction

A conceptually appealing framework for simultaneous stabilization and optimization of feedback systems consists in requiring that the Bellman's optimal value function be in the same time a Lyapunov function. This has been elegantly achieved in Kalman's linear regulator theory as a culmination of earlier efforts by Lurie, Krasovski, Bellman, and many others. However, in dealing with nonlinear problems, the Lyapunov-Bellman concept has serious drawbacks. One of them, the notorious "curse of dimensionality," is frustrating to practitioners. Another one, the question of existence and differentiability of the optimal value function, disturbs the analytically minded. Similar difficulties appear on the Lyapunov side because of the lack of general methods for constructing Lyapunov functions. Nevertheless, the optimum stabilization continues to be one of the fertile concepts stimulating the development of numerical and analytical methods for nonlinear regulator design [4-7]. Most analytical methods assume that the linear part of the system is dominant and design a linear regulator as a first approximation, to be subsequently corrected by series expansions [5,7]. This approach is applicable to many nonlinear systems, but it also has important limitations. First, it is not directly applicable if the linear part is not dominant, second, calculation of expansions increases the dimensionality difficulties, and, third, ill-conditioning due to fast and slow phenomena remains.

The two-time-scale approach presented in this paper avoids linearization and directly addresses the dimensionality and ill-conditioning difficulties. Its philosophy can simply be stated as follows: "Design the slow subsystem first, by assuming that the fast subsystem has already reached its steady state. Then design the fast subsystem for a set of constant values

of the states of the slow subsystem. Combine the two designs by guaranteeing stability and near-optimality properties of the resulting system ." The method proposed in [3] and developed here implements this design philosophy on the systems nonlinear in slow variables and linear in fast variables and control.

The class of systems considered is assumed to be in the standard singular perturbation form exhibiting explicitly a parameter μ, which can be interpreted as the order of magnitude of the ratio of the slow and fast state speeds. Although this form simplifies the definition of the subsystems, the paper does not require any familiarity with singular perturbation techniques. The slow and fast subsystems can be considered as postulates whose validity is subsequently demonstrated by the properties of the actual system controlled by the proposed composite control. Since the proofs of these properties are elementary and make use of only Bellman's principle of optimality and Lyapunov-type arguments, the paper can be read with no more than a basic background in control theory. The steps of the design procedure are presented on a simple example. The method of this paper is radically different from the finite interval trajectory optimization results of [8,9] because of the stability and boundedness requirements fundamental in infinite time problems, which require feedback solutions.

2. Full Problem

The problem considered is to optimally control the nonlinear system

$$\dot{x} = a_1(x) + A_1(x)z + B_1(x)u, \qquad x(0) = x_o \tag{2.1a}$$

$$\mu\dot{z} = a_2(x) + A_2(x)z + B_2(x)u, \qquad z(0) = z_o \tag{.1b}$$

with respect to the cost function

$$J = \int_0^\infty [p(x) + s'(x)z + z'Q(x)z + u'R(x)u]dt \tag{2.2}$$

where $\mu > 0$ is the singular perturbation parameter, x, z are n-, m-dimensional states, respectively, u is an r-dimensional control and the prime denotes a transpose. Regulator problems where the system is linear in the control and nonlinear in the state have been considered earlier [6]. Here the system is also linear in the fast state variable z, as is for example, the case with models of dc motors and synchronous machines [2]. We make an assumption which in addition to differentiability and positivity properties of terms in (2.1), (2.2) also guarantees that the origin is the desired equilibrium.

Assumption I: There exists a domain $D \subset R^n$, containing the origin as an interior point, such that for all $x \in D$ functions a_1, a_2, A_1, A_2, A_2^{-1}, B_1, B_2, p, s, R, and Q are differentiable with respect to x; a_1, a_2, p, and s are zero only at $x = 0$; Q and R are positive definite matrices for all $x \in D$; the scalar $p + s'z + z'Qz$ is a positive definite function of its arguments x and z, that is, it is positive except for $x = 0$, $z = 0$ where it is zero.

An approach to the full problem (2.1), (2.2) would be to assume that a differentiale optimal value function $V(x,z,\mu)$ exists satisfying Bellman's principle of optimality

$$0 = \min_u [p + s'z + z'Qz + u'Ru + V_x(a_1 + A_1z + B_1u) + \frac{1}{\mu} V_z(a_2 + A_2z + B_2u)] \tag{2.3}$$

where V_x, V_z denote the partial derivatives of V. Since the control minimizing (2.3) is

$$u_m = -\frac{1}{2} R^{-1}(B_1'V_x' + \frac{1}{\mu} B_2'V_z'), \tag{2.4}$$

the problem would consist in solving the Hamilton-Jacobi equation

$$0 = p + x'z + z'Qz + V_x(z_1 + A_1 z) + \frac{1}{\mu} V_z(a_2 + A_2 z) - \frac{1}{4}(V_x B_1 + \frac{1}{\mu} V_z B_2)R^{-1}(B_1'V_x' + \frac{1}{\mu} B_2'V_z'),$$

$$V(0,0,\mu) = 0. \tag{2.5}$$

This would be a difficult task even for well behaved nonlinear systems. Due to the presence of $\frac{1}{\mu}$ terms in (2.5), the difficulties with singularly perturbed systems (2.1) increase. The method of this paper avoids these difficulties. In contrast we take advantage of the fact that as $\mu \to 0$ the slow and the fast phenomena in (2.1) separate. We do not deal with the problem (2.1), (2.5) directly. Instead we define two separate lower *dimensional subproblems*, slow and fast. The assumption about existence and differentiability of the optimal value function is then made only for the slow subproblem, while the assumption for the fast subproblem is similar to those made for linear quadratic problems. The solutions of the two subproblems are combined into a composite control whose stabilizing and near optimal properties are the main subject of the paper.

3. Slow Subproblem

Because of the presence of μ, system (2.1) exhibits a "boundary layer," that is, a fast transient in the variable z, after whose decay both x and z vary slowly with time. Setting $\mu = 0$ the fast transient is neglected, that is,

$$\dot{x}_s = a_1(x_s) + A_1(x_s)z_s + B_1(x_s)u_s, \qquad x_s(0) = x_o \tag{3.1a}$$

$$0 = a_2(x_s) + A_2(x_s)z_s + B_2(x_s)u_s, \tag{3.1b}$$

and, since A_2^{-1} is assumed to exist,

$$z_s(x_s) = -A_2^{-1}(a_2 + B_2u_s) \tag{3.2}$$

is eliminated from (3.1a) and (2.2). Then the slow subproblem is to optimally control the slow subsystem

$$\dot{x}_s = a_o(x_s) + B_o(x_s)u, \qquad x_s(0) = x_o \tag{3.3}$$

with respect to

$$J_s = \int_0^\infty [p_o(x_s) + 2s_o'(x_s)u_s + u_s'R_o(x_s)u_s]dt \tag{3.4}$$

where

$$\begin{aligned}
a_o &= a_1 - A_1A_2^{-1}a_2 \\
B_o &= B_1 - A_1A_2^{-1}B_2 \\
p_o &= p - s'A_2^{-1}a_2 + a_2'A_2'^{-1}QA_2^{-1}a_2 \\
s_o &= B_2'A_2'^{-1}(QA_2^{-1}a_2 - \tfrac{1}{2}s) \\
R_o &= R + B_2'A_2'^{-1}QA_2^{-1}B_2.
\end{aligned} \tag{3.5}$$

We note that $x_s = 0$ is the desired equilibrium of the slow subsystem (3.3) for all $x_s \in D$, since, in view of Assumption I, $a_o(0) = 0$ and the integrand in (3.4) is positive definite in x_s and u_s, that is

$$p_o(x_s) + 2s_o'(x_s)u_s + u_s'R_o(x_s)u_s > 0, \qquad x_s \neq 0,\ u_s \neq 0. \tag{3.6}$$

Our crucial Assumption II concerns the existence of the optimal value function $L(x_s)$ for the slow subproblem satisfying the optimality principle

$$0 = \min_{u_s}[p_o(x_s) + 2s_o'(x_s)u_s + u_s'R_o(x_s)u_s + L_x(a_o(x_s)+B_o(x_s)u_s)] \tag{3.7}$$

where L_x denotes the derivative of L with respect to its argument x_s. The elimination of the minimizing control

$$u_s = -R_o^{-1}(s_o + \frac{1}{2} B_o'L_x') \tag{3.8}$$

from (3.7) results in the Hamilton-Jacobi equation

$$0 = (p_o - s_o'R_o^{-1}s_o) + L_x(a_o - B_oR_o^{-1}s_o) - \frac{1}{4} L_xB_oR_o^{-1}B_o'L_x', \quad L(0) = 0, \tag{3.9}$$

where, due to (3.6), $p_o - s_o'R_o^{-1}s_o$ is positive definite in D.

<u>Assumption II</u>: For all $x_s \in D$ equation (3.9) has a unique differentiable positive definite solution $L(x_s)$ with the property that positive constants k_1, k_2, k_3, k_4 exist such that

$$k_1 L_x L_x' \le -L_x \bar{a}_o \le k_2 L_x L_x' \tag{3.10}$$

$$k_3 \bar{a}_o' \bar{a}_o \le -L_x \bar{a}_o \le k_4 \bar{a}_o' \bar{a}_o . \tag{3.11}$$

Assumption II allows $L(x_s)$ to be used as a Lyapunov function guaranteeing the asymptotic stability of $x_s = 0$ for the slow subsystem (3.3) controlled by (3.8), that is for the feedback system

$$\dot{x}_s = a_o - B_oR_o^{-1}(s_o + \frac{1}{2} B_o'L_x') = \bar{a}_o(x_s). \tag{3.12}$$

It also guarantees that D belongs to the region of attraction of $x_s = 0$. For convenience we will take a level surface $L(x_s) = c_o$ to be the boundary of D. It is pointed out that Assumption II does not guarantee the exponential stability. This would be unnecessarily restrictive and would exclude some common slow subsystems such as $\dot{x}_s = -x_s^3$.

Conditions (3.10), (3.11) characterize the slow subproblem solution L by bounding the rate $\dot{L} = L_x \bar{a}_o$ at which it decays to zero along the trajectories of (3.12). These bounds encompass a larger class of nonlinear systems than

do some more common conditions based on exponential stability of linearized models [5,7]. When the solution L of the s..ow subproblem is known, conditions (3.10), (3.11) are readily verifiable. This is how they are used in our two stage design. We first solve the slow subproblem by one of the existing methods, taking advantage of the fact that its dimensionality is lower than that of the full problem. At the end of this stage L is known and (3.10), (3.11) are checked. If they are satisfied, we proceed to the second stage, that is we solve the fast subproblem.

4. Fast Subproblem

To motivate the formulation of the fast subproblem we observe that x being predominantly slow means that only an $O(\mu)$ error is made by replacing x with x_s, or vice versa. Thus, when we subtract (3.1b) from (2.1b) we obtain the system

$$\mu(\dot{z}-\dot{z}_s) = A_2(x)(z-z_s) + B_2(x)(u-u_s) - \mu\dot{z}_s \tag{4.1}$$

which can be further simplified by neglecting the r.h.s. $O(\mu)$ term $-\mu\dot{z}_s$. Defining $z_f = z-z_s$ and $u_f = u-u_s$ the system (4.1) becomes

$$\mu\dot{z}_f = A_2(x)z_f + B_2(x)u_f, \quad z_f(0) = z_o - z_s(0). \tag{4.2}$$

Following a similar reasoning we define

$$J_f = \int_0^\infty (z_f'Q(x)z_f + u_f'R(x)u_f)dt. \tag{4.3}$$

Now (4.2) and (4.3) constitute our fast subproblem for each fixed $x \in D$. It has the familiar linear quadratic form.

Assumption III: For every fixed $x \in D$

$$\mathrm{rank}[B_2,\ A_2B_2,\ldots,A_2^{m-1}B_2] = m. \tag{4.4}$$

Alternatively a less demanding stabilizability assumption can be made. Recalling also that $R(x) > 0$, $Q(x) > 0$ (see Assumption I), we obtain, for each $x \in D$, the optimal solution of the fast subproblem

$$u_f(z_f,x) = -R^{-1}(x)B_2'(x)K(x)z_f \tag{4.5}$$

where $K(x)$ is the positive definite solution of the x-dependent Riccati equation

$$0 = KA_2 + A_2'K - KB_2R^{-1}B_2'K + Q. \tag{4.6}$$

The control (4.5) is stabilizing in the sense that the fast feedback system

$$\mu\dot{z}_f = (A_2-B_2R^{-1}B_2'K)z_f \triangleq \bar{A}_2(x)z_f \tag{4.7a}$$

has the property that

$$\mathrm{Re}\lambda[\bar{A}_2(x)] < 0, \qquad \forall x \in D. \tag{4.7b}$$

5. The Composite Control

Compared to the full problem (2.1)-(2.5), the subproblems are easier to solve due to the fact that the fast subproblem, although parameter dependent, is a linear regulator problem and the slow subproblem, although nonlinear, is of a lower order than the full problem. However, the controls u_s and u_f are applicable to the slow and the fast subsystems, respectively, which do not exist in reality. Our goal is to use u_s and u_f to control the actual full system (2.1). To accomplish this we now form a 'composite' control $u_c = u_s+u_f$, in which x_s is replaced by x, and z_f by $z + A_2^{-1}(a_2+B_2u_s(x))$. Thus the composite control is

$$u_c(x,z) = u_s(x) - R^{-1}B_2'K(z + A_2^{-1}(a_2 - B_2u_s(x)))$$
$$= -R_o^{-1}(s_o + \frac{1}{2}B_o'L_s') - R^{-1}B_2'K(z + \bar{A}_2^{-1}\bar{a}_2) \tag{5.1}$$

where

$$\bar{a}_2(x) = a_2 - \frac{1}{2}B_2R^{-1}(B_1'L_x' + B_2'V_1), \qquad \bar{a}_2(0) = 0$$
$$V_1' = -(s' + 2a_2'K + L_x\bar{A}_1)\bar{A}_2^{-1}$$
$$\bar{A}_1 = A_1 - B_1R^{-1}B_2'K. \tag{5.2}$$

Note that u_c is independent of μ, which simplifies the design procedure when μ is a small but unknown parameter.

For u_c to be a meaningful feedback control of the system (2.1), it must first of all be a stabilizing control. Furthermore for u_c to be a candidate for the optimization of (2.2), the full system (2.1) controlled by u_c must result in a bounded cost (2.2). As $\mu \to 0$, the full cost should approach the cost of the slow subproblem. This would imply that u_c is a near-optimal control and that the regulator problem is well-posed. The boundedness and near-optimality results in the subsequent sections are new, while the stability result is essentially the same as [3], but in a new simpler form.

6. Stability

The full system (2.1) controlled by the composite control (3.1) is

$$\dot{x} = a_1 + A_1z + B_1u_c = \bar{a}_1(x) + \bar{A}_1(x)z, \qquad x(0) = x_o$$
$$\mu\dot{z} = a_2 + A_2z + B_2u_c = \bar{a}_2(x) + \bar{A}_2(x)z, \qquad z(0) = z_o \tag{6.1}$$

where

$$\bar{a}_1 = a_1 - \frac{1}{2}B_1R^{-1}(B_1'L_x' + B_2'V_1), \qquad \bar{a}_1(0) = 0, \tag{6.2}$$

and has the following stability property.

Theorem 6.1: If Assumptions I-III are satisfied, there exists a $\mu^* > 0$ such that the equilibrium $x = 0$, $z = 0$ of system (6.1) is asymptotically stable for all $\mu \in (0, \mu^*]$.

Proof: Introducing

$$z_f = z + \bar{A}_2^{-1}\bar{a}_2, \qquad z_f(0) = z_o + \bar{A}_2^{-1}(x_o)\bar{a}_2(x_o) = z_{fo} \tag{6.3}$$

and $F(x) = (\bar{A}_2^{-1}\bar{a}_2)_x$, we rewrite (6.1) as

$$\dot{x} = \bar{a}_o + \bar{A}_1 z_f, \tag{6.4a}$$

$$\mu\dot{z}_f = \mu F(x)\bar{a}_o + (\bar{A}_2 + \mu F(x)\bar{A}_1)z_f. \tag{6.4b}$$

Observing that (6.4a) has the form of the slow subsystem (3.12) with the additional forcing term $\bar{A}_1 z_f$ and that (6.4b) is an $O(\mu)$ perturbation of the fast subsystem (4.2) controlled by the fast control u_f (4.5), that is of (4.7a), we use the sum of the slow and the fast Lyapunov functions

$$\nu(x, z_f, \mu) = L(x) + \alpha\mu z_f' K(x) z_f \tag{6.5}$$

as a tentative Lyapunov function for (6.4) where α is a positive scalar to be chosen. Since $L(x) > 0$ and $K(x) > 0$ in D, ν is positive definite for all $x \in D$, $z_f \in R^m$ and $\mu > 0$. The proof consists in showing that the time derivative $\dot{\nu}$ of ν with respect to (6.4) is negative definite. After completing the squares $\dot{\nu}$ can be put in the form

$$\dot{\nu} = -g(x,\mu) - \frac{1}{2}\alpha\zeta' Q(x)\zeta - \alpha z_f' M(x, z_f, \mu) z_f \tag{6.6}$$

where

$$\begin{aligned}
g &= -L_x\bar{a}_o - y'Q^{-1}y/2\alpha \\
y &= \bar{A}_1'L_x' + 2\alpha\mu KF\bar{a}_o \\
\zeta &= z_f - Q^{-1}y/\alpha \\
M &= Q/2 + KB_2R^{-1}B_2'K - \mu(KF\bar{A}_1 + \bar{A}_1'F'K) - \mu\dot{K}.
\end{aligned} \tag{6.7}$$

Using the fact that x-dependent quantities in g are bounded for $x \in D$, that is,

$$\|\bar{A}_1Q^{-1}\bar{A}_1'\| \le k_5, \quad \|\bar{A}_1Q^{-1}KF\| \le k_6, \quad 4\|F'KQ^{-1}KF\| \le k_7, \tag{6.8}$$

and recalling that $k_1L_xL_x' \le -L_x\bar{a}_o$, $k_3\bar{a}_o'\bar{a}_o \le -L_x\bar{a}_o$, see (3.10),(3.11), we obtain

$$y'Q^{-1}y \le (k_5 + 3\alpha\mu k_6)L_xL_x' + (3\alpha\mu k_6 + \alpha^2\mu^2k_7)\bar{a}_o'\bar{a}_o \le -\sigma L_x\bar{a}_o \tag{6.9}$$

where

$$\sigma(\alpha\mu) = k_1^{-1}(k_5 + 3\alpha\mu k_6) + k_3^{-1}(3\alpha\mu k_6 + \alpha^2\mu^2k_7). \tag{6.10}$$

It follows from (6.9) that

$$g \ge -L_x\bar{a}_o(1-\sigma/2\alpha) \tag{6.11}$$

and hence, to make g positive definite, it is sufficient to choose $\alpha > \sigma/2$. A convenient choice is to take α to be the value of σ when $\alpha\mu = 1$. Since σ is a monotonically increasing function of $\alpha\mu \ge 0$, this choice implies that

$$g \ge -\frac{1}{2}L_x\bar{a}_o > 0 \qquad \forall\mu \in (0, \tfrac{1}{\alpha}]. \tag{6.12}$$

To complete the proof we need to show that M is also positive definite. Noting that the first two terms of M are positive definite we now establish that they dominate the last two terms, which are small for μ sufficiently small. Using the bounds (6.8) and

$$|\dot{K}| = |K_x \dot{x}| \le |K_x||\bar{a}_o + \bar{A}_1 z_f| \tag{6.13}$$

we conclude that there exist positive constants μ_1 and k_8 such that

$$M \ge \frac{1}{4}(Q + KB_2R^{-1}B_2'K) \tag{6.14}$$

holds for all $x \in D$, all z_f such that $|z_f| \le k_8$, and all $\mu \in (0,\mu_1]$. Thus for all

$$\mu \in (0,\mu^*], \qquad \mu^* = \min(\frac{1}{\alpha}, \mu_1) \tag{6.15}$$

the derivative $\dot{\nu}$ of ν in (6.5) for system (6.1), or, equivalently, for system (6.4), is negative definite and hence the equilibrium $x = 0$, $z = 0$, is asymptotically stable.

From this proof we can readily obtain an estimate of the region of attraction of $x = 0$, $z = 0$. A well known estimate is the set of points x, z encompassed by the largest closed surface $\nu(x,z,\mu) = c^*$ for which $\dot{\nu}$ is negative definite. To each fixed $\mu \in (0,\mu^*]$ there corresponds one such set denoted by S_μ. All S_μ sets contain all $x \in D$, but differ in the magnitudes of z, because, as it can be inferred from the above proof, the larger μ is, the smaller z_f is allowed. Thus the set corresponding to the largest value of μ, that is to μ^*, is the largest set and is denoted by S^*. Since this set is the intersection of all S_μ sets, it can serve as a common estimate for the regions of attraction for all values of $\mu \in (0,\mu^*]$. A proof of this fact consists of the calculations analogous to those leading to (6.6) through (6.15), but this time for ν with μ fixed at $\mu = \mu^*$, that is for $\nu(x,z,\mu^*)$, rather than for $\nu(x,z,\mu)$. Omitting these calculations we state the result in the form useful for our subsequent analysis.

Corollary 6.2: Under the assumptions of Theorem 6.1 there exist positive constants μ^* and c^* such that the set

$$S^*(x,z) = \{x,z: \ \nu(x,z,\mu^*) \le c^*\} \tag{6.16}$$

belongs to the region of attraction of $x=0$, $z=0$ for all $\mu \in (0,\mu^*]$, that is all trajectories of (6.1) originating in S^* at $t=0$ remain in S^* for all $t>0$ and converge to $x=0$, $z=0$, as $t\to\infty$.

7. Boundedness of J

Asymptotic stability of an equilibrium at the origin is not sufficient to guarantee that an integral of the type (2.2) will be finite along the trajectories asymptotically converging to this equilibrium. For example, when the control $u=-x^2-x^5$ is applied to the system $\dot{x}=x^2+u$, then the equilibrium $x=0$ of $\dot{x}=-x^5$ is asymptotically stable. However the solutions for $x(0)=x_o \neq 0$ are

$$x(t) = \mathrm{sign}(x_o)(4t+(x_o)^{-4})^{-1/4}, \tag{7.1}$$

and hence the cost

$$J = \int_o^\infty (x^4 + 1/2\ u^2)dt \tag{7.2}$$

is infinite. Thus it is not sufficient that our composite control be only a stabilizing control. To qualify as a candidate for near-optimality u_c must also produce a bounded J. To show that this is the case we use the following lemma from [1], which is implicit in [4,6].

Lemma 7.1: Suppose that system (2.1) controlled by $u(x,z)$ has $x=0$, $z=0$ as its asumptotically stable equilibrium for all $x_o, z_o \in S$. Let this fact be established by a positive definite Lyapunov function $q(x,z)$, whose derivative

$\dot{q}(x,z)$ is negative definite in S. If there exists a ball β centered at $x=0$, $z=0$ such that for all $x,z \in \beta$,

$$p + s'z + z'Qz + u'Ru + \dot{q} \leq 0, \tag{7.3}$$

then the cost (2.2) is finite along all the trajectories which originate in S and is bounded from above by q.

Proof: Let t_β be the instant when a trajectory τ originating from $x_o, z_o \in S$ enters the ball β through x_β, z_β for the last time and stays in β thereafter. The part of the cost along τ over the finite interval $[0,t_\beta]$ is obviously finite. Denoting the remaining part of the cost over (t_β,∞) by J_β and integrating (7.3) from t_β to ∞ we obtain

$$J_\beta + [q(0,0) - q(x_\beta,z_\beta)] \leq 0 \tag{7.4}$$

which in view of $q(0,0) = 0$ and the fact that $q(x_\beta,z_\beta)$ is finite, proves that J_β is bounded.

To apply this lemma we substitute (5.1) and (6.3) for u_c and z, respectively into

$$J_c = \int_o^\infty (p + s'z + z'Qz + u_c'Ru_c)dt = \int_o^\infty f_c(x,z)dt \tag{7.5}$$

and rewrite the integrand as

$$f_c(x,z) = -L_x\bar{a}_o - s_1'z_f + z_f'(Q + KB_2R^{-1}B_2'K)z_f = f(x,z_f) \tag{7.6}$$

where

$$s_1 = s + KB_2R^{-1}(B_1'L_x' + B_2'V_1) + 2(Q + KB_2R^{-1}B'_2K)\bar{A}_2^{-1}\bar{a}_2. \tag{7.7}$$

It is important to note that the dependence on z_f in (7.6) is indicated explicitly, that is, the term $L_x\bar{a}_o$ is independent of z_f. Furthermore, $f(x, -z_f) > 0$ because $f(x,z_f) > 0$ for all $x \in D$ and $z_f \in R^m$, $x \neq 0$, $z_f \neq 0$.

Theorem 7.2: Under Assumptions I-III, the composite control u_c produces a cost J_c which is bounded from above by 4ν for all $\mu \in (0,\mu^*]$.

Proof: From (6.12) and (6.15) we obtain

$$f(x,z_f) + 4\dot{\nu} \leq -f(x, -z_f) \leq 0. \tag{7.8}$$

From Theorem 6.1 we know that 4ν is a Lyapunov function for system (6.4) and we use it as q in Lemma 7.1, which in view of (7.4) completes the proof.

8. Near Optimality

The question can now be posed whether u_c, being a stabilizing control which produces a bounded cost, is also near optimal in the sense that as $\mu \to 0$ the cost J_c tends to the optimal cost for $\mu = 0$, that is the optimal cost L(x) of the reduced problem. This question is answered by expressing J_c as

$$J_c(z,x,\mu) = L(x) + \mu V_1'(x)z + \mu z'K(x)z + \mu J_4(x,z,\mu) \tag{8.1}$$

where the first two μ-terms are suggested by the linear-quadratic form of the fast subproblem. If we prove that J_4 remains bounded as $\mu \to 0$, this will guarantee that $J_c(x,z,\mu) \to L(x)$.

Theorem 8.1: Under Assumptions I-III, the composite control produces cost (8.1) in which J_4 remains bounded as $\mu \to 0$.

Proof: Cost $J_c(x,z,\mu)$ of system (2.1) controlled by u_c satisfies partial differential equation

$$\begin{aligned} p + s'z + z'Qz + u_c'Ru_c + (J_c)_x(a_1 + A_1 z + B_1 u_c) \\ + (J_c)_z(a_2 + A_2 z + B_2 u_c)/\mu = 0, \end{aligned} \tag{8.2}$$

$$J_c(0,0,\mu) = 0.$$

We have shown in [3] that the substitution of (8.1) into (8.2) and the use of (3.9), (4.6), and (5.2), reduce (8.2) to

$$J_{4x}(\bar{a}_1 + \bar{A}_1 z) + \frac{1}{\mu} J_{4z}(\bar{a}_2 + \bar{A}_2 z) = -(V_1' z + z' K z)_x (\bar{a}_1 + \bar{A}_1 z),$$
$$J_4(0,0,\mu) = 0. \tag{8.3}$$

This expression, and the fact following from Theorem 7.1 that μJ_4 is bounded, are used in the Appendix to complete the proof.

In addition to the near optimality of the composite control, Theorem 8.1 also shows that the full regulator problem is well posed in the sense that the same cost results from neglecting μ in the system model and then applying the control u_s to (3.3), or first applying the control u_c to (2.1) and then neglecting μ.

9. Two Stage Design

The steps of the proposed two stage design will be presented on a simple example of the system

$$\dot{x} = -\frac{3}{4} x^3 + z \tag{9.1a}$$

$$\mu \dot{z} = -z + u \tag{9.1b}$$

and the cost functional

$$J = \int_0^\infty (x^6 + \frac{3}{4} z^2 + \frac{1}{4} u^2) dt. \tag{9.2}$$

Step 1: The slow subproblem

$$\dot{x}_s = -\frac{3}{4} x_s^3 + u_s \tag{9.3}$$

$$J_s = \int_0^\infty (x_s^5 + u_s^2) dt \tag{9.4}$$

consists in solving the Hamilton-Jacobi equation

$$L_x = \frac{dL}{dx_s} = x_s^3, \qquad L(0) = 0 \tag{9.5}$$

which yields

$$L = \frac{1}{4} x_s^4, \qquad u_s = -\frac{1}{2} x_s^3, \qquad \dot{x}_s = -\frac{5}{4} x_s^3.$$

<u>Step 2</u>: Testing the conditions (3.10), (3.11)

$$k_1 x_s^5 \le \frac{5}{4} x_5^6 \le k_2 x_s^6, \tag{9.7}$$

$$\frac{25}{16} k_3 x_s^6 \le \frac{5}{4} x_s^6 \le \frac{25}{16} k_4 x_s^6, \tag{9.8}$$

we see that they are satisfied by

$$k_1 = k_2 = \frac{5}{4}, \qquad k_3 = k_4 = \frac{4}{5}. \tag{9.9}$$

<u>Step 3</u>: The fast subproblem

$$\mu \dot{z}_f = -z_f + u_f \tag{9.10}$$

$$J_f = \int_0^\infty \left(\frac{3}{4} z_f^2 + \frac{1}{4} u_f^2\right) dt \tag{9.11}$$

is in this case independent of x and its solution is

$$K = \frac{1}{4}, \qquad u_f = -z_f, \qquad \mu \dot{z}_f = -2 z_f. \tag{9.12}$$

<u>Step 4</u>: The design is completed by forming the composite control

$$u_c = -x^3 - z \tag{9.13}$$

and applying it to the full system (9.1). The final feedback system (6.1) is

$$\dot{x} = -\frac{3}{4} x^3 + z \tag{9.14a}$$

$$\mu \dot{z} = -x^3 - 2z. \tag{9.14b}$$

It should be noted that this system could not have been designed by methods based on linearization, since its linearized model at $x = 0$, $z = 0$ has a zero eigenvalue. However, Theorem 6.1 guarantees that the equilibrium $x = 0$, $z = 0$ is asymptotically stable for μ sufficiently small.

Step 5: With the help of Theorem 6.1 and Corollary 6.2 we can further analyze stability properties of the designed system (9.14) which is first transformed by $z_f = z + \frac{1}{2} x^3$ into (6.4), that is into

$$\dot{x} = -\frac{5}{4} x^3 + z_f \tag{9.17a}$$

$$\mu \dot{z}_f = -\mu \frac{15}{8} x^5 - (2 - \mu \frac{3}{2} x^2) z_f . \tag{9.17b}$$

The Lyapunov function (6.5) is

$$\nu = \frac{1}{4} x^4 + \alpha\mu \frac{1}{4} z_f^2 \tag{9.18}$$

and to analyze its derivative (6.6) we evaluate the bounds (6.8),

$$k_5 \geq \frac{4}{3}, \qquad k_6 \geq \frac{4}{3} \frac{1}{4} \frac{3}{2} x^2, \qquad k_7 \geq 4 \frac{9x^4}{4} \frac{1}{16} \frac{4}{3}. \tag{9.19}$$

They are to be used to find an α guarateeing that g in (6.7) is positive definite for all $x \in D$. In this example the choice of D is free, since the slow subsystem is asymptotically stable in the large. Suppose that we are interested in $x \in [0, \frac{1}{2}]$. Then $k_6 \geq \frac{1}{8}$, $k_7 \geq \frac{3}{64}$ and α is obtained from (6.10) as $\alpha = \sigma(1)$, that is

$$\alpha = \frac{4}{5}(\frac{4}{3} + \frac{3}{8}) + \frac{5}{4}(\frac{3}{8} + \frac{3}{64}) = \frac{881}{480}. \tag{9.20}$$

With this α it can be easily verified that

$$g = \frac{5}{4} x^6 (1 - \frac{8}{15\alpha} (1 - \frac{15}{16} \alpha\mu x^2)^2) > 0 \tag{9.21}$$

for all $x \in (0, \frac{1}{2}]$ and all $\mu \in (0, \frac{1}{\alpha}]$. Next we find μ_1 such that

$$M = \frac{3}{8} + \frac{1}{4} - \mu \frac{3}{4} x^2 > 0 \tag{9.22}$$

for all $x \in (0, \frac{1}{2}]$ and all $\mu \in (0, \mu_1]$. Clearly $\mu_1 < \frac{10}{3}$ and hence $\mu^* = \frac{1}{\alpha} = \frac{480}{881}$

guarantees that $\dot{v}$ is negative definite for all $\mu \in (0, \mu^*]$, all $x \in [0, \frac{1}{2}]$, and all z_f. We note that in this example there is no bound on z_f because $\dot{K} = 0$ and hence M does not depend on z_f. In general a bound on z_f would be required for positive definiteness of M. It should also be noted that for a different set of x, a different μ^* would be obtained. The presented sequence of conditions for $\dot{v} < 0$ is convenient when μ is a parameter at the designer's disposal. When μ is a fixed physical parameter, an alternative treatment of (9.20), (9.21), and (9.22) starting with μ given, would determine the allowed x and, in general, z_f.

Conclusion

The proposed composite control circumvents the dimensionality and conditioning difficulties and takes advantage of the two time scale behavior of the considered class of nonlinear systems. In spite of the singularly perturbed form (2.1), these systems need not be singularly perturbed, that is μ need not be small. Among the results of this paper are the specific bounds on μ, which, as the example shows, can be 0.5 or larger. Estimates of the region of stability are given which depend on μ, but not on the assumption that $\mu \to 0$. The only result that remains restricted to $\mu \to 0$ is near optimality. It is conceivable that by a similar development bounds on the performance loss can be obtained. Another improvement is likely in relaxing conditions (3.10), (3.11). There exist successful applications of the composite control when (3.10), (3.11) are not satisfied. Nonetheless (3.10), (3.11) are less restrictive than exponential stability conditions based on linearization. In the first stage of the two-stage design the lower order nonlinear slow subproblem needs to be solved. It would be of interest to develop a numerical method whereby along the slow

solution also the local values of the fast subproblem matrix K(x) would be generated. Finally, the assumption that the fast variables appear linearly avoids technical complications, but is not crucial for the applicability of the two time scale approach. Extensions to broader classes of systems are possible.

Appendix

We complete the proof of Theorem 8.1 by first rewriting J as

$$\begin{aligned} J_c &= L + \mu V_1'(z_f + z_s) + \mu(z_f + z_s)'K(z_f + z_s) + \mu J_4 \\ &= L + \mu\bar{V}_1' z_f + \mu z_f' K z_f + \mu\bar{J}_4 \end{aligned} \tag{A1}$$

where

$$\begin{aligned} &z = z_f + z_s, \qquad z_s = -\bar{A}_2^{-1}(x)\bar{a}_2(x) \\ &\bar{V}_1 = V_1 + 2Kz_s, \qquad \bar{J}_4 = J_4 + V_1'z_s + z_s'Kz_s. \end{aligned} \tag{A2}$$

With respect to system (6.4) J_c satisfies the partial differential equation

$$f(x,z_f) + (J_c)_x(\bar{a}_o + \bar{A}_1 z_f) + (J_c)_z(\mu F\bar{a}_o + (\bar{A}_2 + \mu F\bar{A}_1)z_f) = 0 \tag{A3}$$

where f is given in (7.6). Taking the partials of J_c in (A1) and substituting into (A3), we obtain

$$(\bar{J}_4)_x\dot{x} + (\bar{J}_4)_z\dot{z}_f = -(z_f'\bar{V}_{1x} + \bar{V}_1'F + 2z_f'KF)(\bar{a}_o + \bar{A}_1 z_f) - z_f'\dot{K}z_f = -f_1. \tag{A4}$$

A further substitution

$$\bar{V}_1 = (\bar{A}_2')^{-1}(s_1 + \bar{A}_1 L_x') \tag{A5}$$

where s_1 is as in (7.7), makes it possible to complete the squares like in (6.6). We thus establish that

$$-f_1 - (z_f'\bar{J}_{2x} + L_x\bar{A}_1'\bar{A}_2^{-1}F + 2z_f'J_3F)(\bar{a}_o + \bar{A}_1 z_f) + z_f'(J_{3x}(\bar{a}_o + \bar{A}_1 z_f))z_f \tag{A6}$$

is bounded from above by

$$-[(1+c_3)/k_1 + (c_1+c_2)/k_3]L_x\bar{a}_o + (2+c_4)z_f'z_f \tag{A7}$$

and from below by

$$\frac{1}{2}[(1+c_3)/k_2 + (c_1+c_2)/k_4]L_x\bar{a}_o - (2+c_4)z_f'z_f \tag{A8}$$

where

$$c_1 \geq \|H_1'H_1\| \;, \qquad H_1 = \bar{A}_1'\bar{A}_2^{-1}F$$

$$c_2 \geq \|H_2'H_2\| \;, \qquad H_2 = (\bar{V}_{1x} + 2KF)\bar{a}_o$$

$$c_3 \geq \|H_3'H_3\| \;, \qquad H_3 = \bar{A}_1'\bar{A}_2^{-1}F\bar{A}_1 \tag{A9}$$

$$c_4 \geq \|\dot{K} + \frac{1}{2}(\bar{V}_{1x} + 2KF)\bar{A}_1 + \frac{1}{2}\bar{A}_1'(\bar{V}_{1x}' + 2F'K)\| \,.$$

From (6.13) we know that $\dot{K}$, and hence c_4, remain bounded as $\mu \to 0$. Furthermore, rewriting $f(x,z_f) > 0$ in (7.6) as

$$s_1'z_f \geq -L_x\bar{a}_o + z_f'(Q + KB_2R^{-1}B_2')z_f, \tag{A10}$$

and using the fact that the right hand side quantity is positive definite for all $x \in D$, $z_f \in R^m$, we obtain by substituting $\pm\bar{A}_2^{-1}(\bar{a}_o + \bar{A}_1z_f)$ for z_f,

$$\|s_1'\bar{A}_2^{-1}(\bar{a}_o + \bar{A}_1z_f)\| \leq -(1 + 2c_5/k_3)L_x\bar{a}_o + 2c_6z_f'z_f \tag{A11}$$

where

$$c_5 \geq \|N\| = \|(\bar{A}_2^{-1})'(Q + KB_2R^{-1}B_2'K)\bar{A}_2^{-1}\|$$

$$c_6 \geq \|\bar{A}_1'N\bar{A}_1\| \,. \tag{A12}$$

Combining (A6) and (A11) we conclude that there exists $\gamma > 0$ such that f_1 is bounded by $|\gamma\dot{v}|$, which, by Lemma 7.1, proves that

$$J_4 = \int_o^\infty f_1 dt \tag{A13}$$

is bounded.

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SINGULAR PERTURBATION RESULTS FOR A CLASS OF STOCHASTIC CONTROL PROBLEMS*

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Abstract

Composite control orginally proposed in a deterministic context is generalized to the problem with white noise inputs. However, the approach used here is radically different from the deterministic approach. Presence of noise smoothed the system behavior and allowed a more complete solution than in the deterministic case.

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INTRODUCTION

We study in this paper a stochastic version of the problem considered by J. H. Chow and P. Kokotovic [2]. Namely, we consider

$$dx = (c(x)z + d(x) + 2\rho(x)v(t))dt + \sqrt{2}\, dw_1$$

$$dz = \frac{1}{\epsilon}(a(x)z + b(x) + 2\alpha(x)v(t))dt + \sqrt{2}\, dw_2$$

$$x(0) = x, \quad z(0) = z$$

$$J^{\epsilon}_{x,z}(v(\cdot)) = E\int_0^{\infty} e^{-\gamma t}[(f(x) + h(x)z)^2 + v(t)^2]dt.$$

Chow and Kokotovic have considered this problem without driving white noises. It turns out that the introduction of the noises smoothes the system, and allows to obtain a fairly complete solution of the singular perturbation problem, without the assumptions made in the deterministic case. We however assume all function of x sufficiently smooth and bounded, and the discount γ large enough (but fixed).

We write formally the equation of dynamic programming and study its asymptotic expansion. We prove that all the terms of the expansion are uniquely defined and smooth (depending on the smoothness assumptions on the coefficients).

Then as in Chow and Kokotovic we consider a composite control and prove that it maintains the pay off bounded by a constant independent of ϵ. From that it follows that $\inf J^{\epsilon}_{x,z}(v(\cdot))$ remains bounded as $\epsilon \to 0$. It is possible from this estimate to show that the initial equation of dynamic programming has a maximum solution in some Sobolev space with weights (as in Bennoussan-Lions [1]). However we cannot prove a convergence result for the inf. What we prove is that

$$\inf_{V} J^{\varepsilon}_{x,z}(v(\cdot)) \to u^{o}_{o}(x)$$

where V is a more restrictive class of controls (namely those for which $E\int_0^\infty e^{-\gamma t}|z(t)|^2 dt < M$, where M is a constant independent of ε). Moreover $u^o_o(x)$ is the 1st term of the expansion.

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Contents

1. Setting of the Model

Let us consider functions a(x), b(x), c(x), d(x), $\alpha(x)$, $\beta(x)$ satisfying

$$a, b, c, d, \alpha, \beta \text{ smooth and bounded, } \alpha \neq 0, \ a \neq 0. \tag{1.1}$$

Let $w_1(t)$, $w_2(t)$ be two Wiener processes, scalar, standard and independent one from each other.

We consider the stochastic system of the equations

$$dx = (c(x)z + d(x) + 2\beta(x)v(t))dt + \sqrt{2}\, dw_1$$

$$dz = \frac{1}{\epsilon}(a(x)z + b(x) + 2\alpha(x)v(t))dt + \sqrt{2}\, dw_2 \tag{1.2}$$

$$x(0) = x \qquad\qquad z(0) = z.$$

The control $v(\cdot)$ is a non anticipative process such that $E \int_0^\infty e^{-\gamma t}|v(t)|^2 dt < \infty$. We consider the payoff

$$J^\epsilon_{x,z}(v(\cdot)) = E \int_0^\infty e^{-\gamma t}[(f(x) + h(x)z)^2 + v(t)^2]dt \tag{1.3}$$

where

$$\gamma > 0 \text{ constant} \tag{1.4}$$

$$f,\ h \text{ bounded smooth functions} \tag{1.5}$$

we are interested in the behavior as $\epsilon \to 0$, of the Bellman function

$$u^\epsilon(x,z) = \inf_{v(\cdot)} J^\epsilon_{x,z}(v(\cdot)) \tag{1.6}$$

Formally we can write the Bellman equation which is satisfied by u^ϵ. Namely

$$-\Delta u^\epsilon + \gamma u^\epsilon = (f(x) + h(x)z)^2 + \inf_v [v^2 + u^\epsilon_x\, 2\beta v + \frac{u^\epsilon_z}{\epsilon}\, 2\alpha v] + $$
$$+ u^\epsilon_x(cz + d) + \frac{u^\epsilon_z}{\epsilon}(az + b)$$

or

$$-\Delta u^\epsilon + \gamma u^\epsilon + (\beta u^\epsilon_x + \frac{\alpha u^\epsilon_z}{\epsilon})^2 - u^\epsilon_x(cz + d) - \frac{u^\epsilon_z}{\epsilon}(az + b) = (f + hz)^2 \tag{1.7}$$

The optimal feedback is given by

$$v^\epsilon(x,z) = -\beta(x)u^\epsilon_x(x,z) - \frac{\alpha(x)u^\epsilon_z(x,z)}{\epsilon} \tag{1.8}$$

we will refer to x as the slow system and to z as the fast system. The slow system is strongly nonlinear, the fast system is linear with coefficients depending nonlinearly on x.

We will not study equation (1.7) for general ϵ, it will be used to derive the expansion. Rather we will be interested in considering (1.6) (taking it as a definition of u^ϵ) for ϵ small. We will define a limit problem which will be the stochastic control problem for a reduced system (obtained formally after multiplication by ϵ and setting $\epsilon = 0$ in the equation of the fast system). The stochastic control problem for the reduced system will be solved completely using Bellman equation. Now considering

$$w^\epsilon(x,y) = \inf_{v(\cdot)\in V} J^\epsilon_{x,z}(v(\cdot))$$

where V is a restricted class of control (see (5.6)), namely the class of controls for which the system (slow and fast) respect a growth condition, then w^ϵ will be approximated, up to ϵ by the value function of the reduced system.

2. Formal Expansion

We look for an asymptotics of the following form

$$u^\epsilon(x,z) = \sum_{r=0}^{\infty} \epsilon^p \sum_{\ell=0}^{p+1} z^\ell \, u^p_\ell(x) \tag{2.1}$$

where u^p_ℓ are functions to be identified.

For convenience we define

$$u^p_\ell = 0 \quad \text{for } \ell \geq p+2, \quad \text{and } \ell < 0. \tag{2.2}$$

The following formulas which are easily verified

$$u_x^\epsilon = \sum_{p=0}^{\infty} \epsilon^p \sum_{\ell=0}^{p+1} z^\ell u_{\ell x}^p$$

$$\frac{u_z^\epsilon}{\epsilon} = \sum_{p=-1}^{\infty} \epsilon^p \sum_{\ell=0}^{p+1} (\ell+1) z^\ell u_{\ell+1}^{p+1}$$

$$z u_x^\epsilon = \sum_{p=0}^{\infty} \epsilon^p \sum_{\ell=1}^{p+2} z^\ell u_{\ell-1,x}^p \tag{2.3}$$

$$\frac{z u_z^\epsilon}{\epsilon} = \sum_{p=-1}^{\infty} \epsilon^p \sum_{\ell=1}^{p+2} \ell z^\ell u_\ell^{p+1}$$

$$u_{xx}^\epsilon = \sum_{p=0}^{\infty} \epsilon^p \sum_{\ell=0}^{p+1} z^\ell u_{\ell xx}^p$$

$$u_{zz}^\epsilon = \sum_{p=1}^{\infty} \epsilon^p \sum_{\ell=0}^{p-1} (\ell+1)(\ell+2) z^\ell u_{\ell+2}^p$$

$$\beta u_x^\epsilon + \alpha \frac{u_z^\epsilon}{\epsilon} = \alpha\,\epsilon^{-1} u_1^0 + \sum_{p=0}^{\infty} \epsilon^p \sum_{\ell=0}^{p+1} z^\ell (\beta u_{\ell x}^p + (\ell+1)\alpha u_{\ell+1}^{p+1}) \tag{2.4}$$

$$\begin{aligned}(\beta u_x^\epsilon + \alpha \frac{u_z^\epsilon}{\epsilon})^2 &= \alpha^2 \epsilon^{-2} (u_1^0)^2 + 2u_1^0 \alpha \sum_{p=0}^{\infty} \epsilon^{p-1} \sum_{\ell=0}^{p+1} z^\ell (\beta u_{\ell x}^p + \alpha(\ell+1) u_{\ell+1}^{p+1}) \\ &+ \sum_{p=0}^{\infty} \epsilon^p \sum_{\ell=0}^{p+2} z^\ell \sum_{n=0}^{p} \sum_{j=0 \vee (\ell-p+n-1)}^{\ell \wedge (n+1)} (\beta u_{jx}^n + \alpha(j+1) u_{j+1}^{p+1})(\beta u_{\ell-j,x}^{p-n} \\ &+ \alpha(\ell-j+1) u_{\ell-j+1}^{p-n+1})\end{aligned} \tag{2.5}$$

and are used in equating powers of $\epsilon^p z^\ell$ in (1.7). We remark immediately that

$$u_1^0 = 0 \tag{2.6}$$

We then organize the calculations as follows. Assume that at some stage $p \geq 1$ we know

$$u_{p+1}^p \; u_p^p \; \ldots, \; u_1^p, \quad \text{but not } u_0^p$$

and

$$u_\ell^r \quad \text{for} \quad D \leq r \leq p-1 \quad \ell = r+1, \ldots, 0$$

then we will successively compute

$$u^{p+1}_{p+2},\quad u^{p+1}_{r+1},\ldots,u^{p+1}_{1},\quad u^{p}_{0}$$

The case $p=1$ is slightly particular. We start with it. So we compute successively u^1_2, u^1_1, u^0_0. Taking advantage of convention (2.2) and (2.6) we consider the sums in (2.3) with r running from 0 to ∞ and ℓ from 0 to $p+2$.

Therefore we can write the (p,ℓ) problem $(p\geq 0, 0\leq \ell\leq p+2)$ as follows

$$-u^p_{\ell xx}-(\ell+1)(\ell+2)u^p_{\ell+2}+\gamma u^p_\ell+\sum_{n=0}^{\ell\wedge(n+1)}(\beta u^n_{jx}+\alpha(j+1)u^{n+1}_{j+1})(\beta u^{p-n}_{\ell-j,x}$$
$$+\alpha(\ell-j+1)u^{p-n+1}_{\ell-j+1})-[cu^p_{\ell-1,x}+du^p_{\ell x}+b(\ell+1)u^{p+1}_{\ell+1}+a\ell u^{p+1}_{\ell}]=(\ell^2\chi_{\ell=0}+2fh\chi_{\ell=1}$$
$$+h^2\chi_{\ell=2})\chi_{p=0} \tag{2.7}$$

we apply (2.7) with $p=0$, $\ell=2$, which will permit us to compute u^1_2. We obtain

$$4\alpha^2(u^1_2)^2-2au^1_2=h^2 \tag{2.8}$$

and take

$$u^1_2=\frac{a+\sqrt{a^2+4\alpha^2h^2}}{4\alpha^2} \tag{2.9}$$

We next compute u^1_1 by writing (2.7) with $p=0$, $\ell=1$. We get

$$4\alpha u^1_2(\beta u^0_{0x}+\alpha u^1_1)-(cu^0_{0x}+2bu^1_2+au^1_1)=2fh \tag{2.10}$$

from which we deduce u^1_1 as an affine function of u^0_0, namely (noting $4\alpha^2u^1_2-a=\sqrt{a^2+4\alpha^2h^2}=\Delta$)

$$u^1_1=\frac{u^0_{0x}(c-4\alpha\beta u^1_2)+2fh+2bu^1_2}{\Delta}\ . \tag{2.11}$$

We next obtain the equation for u_0^0. It comes from (2.7) with $p=0$, $\ell=0$. We obtain

$$-u_{0xx}^0 + \gamma u_0^0 + (\beta u_{0x}^0 + \alpha u_1^1)^2 - [du_{0x}^0 + bu_1^1] = f^2 \tag{2.12}$$

But from (2.10) and (2.11) we deduce

$$\beta u_{0x}^0 + \alpha u_1^1 = \frac{u_{0x}^0(\alpha c - \beta a) + \alpha(2bu_2^1 + 2fh)}{\Delta} \tag{2.13}$$

therefore from (2.12)

$$-u_{0xx}^0 + \gamma u_0^0 + \frac{(u_{0x}^0)^2(\alpha c-\beta a)^2}{\Delta^2} + \frac{\alpha^2(2bu_2^1+2fh)^2}{\Delta^2} + \frac{2\alpha}{\Delta^2}(2bu_2^1+2fh)(\alpha c-\beta a)u_{0x}^0 - du_{0x}^0$$

$$-\frac{bu_{0x}^0(c-4\alpha\beta u_2^1)}{\Delta} - \frac{b(2fh+2bu_2^1)}{\Delta} = f^2$$

$$\begin{aligned} -u_{0xx}^0 + \gamma u_0^0 + u_{0x}^0[a(cb-ad) + 4h\alpha(f(\alpha c-\rho a) + h(\beta b - d\alpha))]\frac{\lambda}{a^2+4\alpha^2h^2} \\ + (u_{0x}^0)^2\,\frac{(\alpha c-\rho a)^2}{a^2+4\alpha^2h^2} = \frac{(fa-bh)^2}{a^2+4\alpha^2h^2} \end{aligned} \tag{2.14}$$

Then u_0^0 is solution of a nonlinear elliptic equation which will be studied in the next section. We can now assume $p \geq 1$, we know u_ℓ^r for $0 \leq r \leq p-1$, and

$$u_{p+1}^p \quad u_p^p \ldots u_1^p$$

We compute successively

$$u_{p+2}^{p+1} \quad u_{p+1}^{p+1} \ldots u_1^{p+1} \quad u_0^p.$$

We compute u_{p+2}^{p+1} by considering (2.7) with $p=p$, $\ell=p+2$, we obtain

$$\sum_{n=0}^{p}(\beta u^{n}_{n+1,x}+\alpha(n+2)u^{n+1}_{n+2})(\beta u^{p-n}_{p-n+1,x}+\alpha(p-n+2)u^{p-n+1}_{p-n+2}) - [cu^{p}_{p+1,x}+a(p+2)u^{p+1}_{p+2}] = 0$$

i.e.

$$(p+2)(4\alpha^2 u^1_2 - a)u^{p+1}_{p+2} + (4\alpha\beta u^1_2 - c)u^{p}_{p+1,x} + \sum_{n=1}^{p-1}(\beta u^{n}_{n+1,x}+\alpha(n+2)u^{n+1}_{n+2})(\beta u^{p-n}_{p-n+1,x}$$
$$+\alpha(p-n+2)u^{p-n+1}_{p-n+2}) = 0$$

hence

$$u^{p+1}_{p+2} = [u^{p}_{p+1,x}(c-4\alpha\beta u^1_2) - \sum_{n=1}^{p-1}(\beta u^{n}_{n+1,x}+\alpha(n+2)u^{n+1}_{n+2})(\beta u^{p-n}_{p-n+1,x}$$
$$+ \alpha(p-n+2)u^{p-n+1}_{p-n+2})]\ \frac{1}{(p+2)\Delta} \qquad p \ge 1 \left(\sum_{n=1}^{p-1} = 0 \quad \text{if } p = 1\right) \tag{2.15}$$

Suppose now that we have computed

$$u^{p+1}_{k} \qquad \text{for} \qquad p+2 \ge k \ge \ell+1$$

with $\ell \ge 2$, and we want to compute u^{p+1}_{ℓ}, $\ell \ge 2$. We consider equation (2.7), which we write as follows

$$- u^{p}_{\ell xx} - (\ell+1)(\ell+2)u^{p}_{\ell+2} + \gamma u^{p}_{\ell} + \sum_{n=1}^{p-1}\ \sum_{j=0 \vee (\ell-p+n-1)}^{\ell \wedge (n+1)}(\beta u^{n}_{jx}+\alpha(j+1)u^{n+1}_{j+1})(\beta u^{p-n}_{\ell-j,x}$$
$$+ \alpha(\ell-j+1)u^{p-n+1}_{\ell-j+1}) + 2(\beta u^{p}_{\ell x}+\alpha(\ell+1)u^{p+1}_{\ell+1})(\beta u^{0}_{0x}+\alpha u^1_1) + 4\alpha u^1_2(\beta u^{p}_{\ell-1,x}+\alpha\ell u^{p+1}_{\ell})$$
$$- [cu^{p}_{\ell-1,x}+du^{p}_{\ell,x}+b(\ell+1)u^{p+1}_{\ell+1}+a\ell u^{p+1}_{\ell}] = 0$$

from which we deduce

$$(4\alpha^2 u^1_2 - a)\ell u^{p+1}_{\ell} = (c-4\alpha_\beta u^1_2)u^{p}_{\ell-1,x} + u^{p+1}_{\ell+1}(\ell+1)(b-2\alpha(\beta u^0_{0x}+\alpha u^1_1))$$
$$+ u^{p}_{\ell,x}(d-2\beta(\beta u^0_{0x}+\alpha u^1_1)) + u^{p}_{\ell xx} + (\ell+1)(\ell+2)u^{p}_{\ell+2} - \gamma u^{p}_{\ell} \tag{2.16}$$
$$- \sum_{n=1}^{p-1}\ \sum_{j=0 \vee (\ell-p+n-1)}^{\ell \wedge (n+1)}(\beta u^{n}_{jx}+\alpha(j+1)u^{n+1}_{j+1})(\beta u^{p-n}_{\ell-j,x}+\alpha(\ell-j+1)u^{p-n+1}_{\ell-j+1})$$

It remains to compute u_1^{p+1} and u_0^p.

We write (2.7) for p and $\ell=1$. We obtain

$$- u^p_{1,xx} - 6u^p_3 + \gamma u^p_1 + \sum_{n=1}^{p-1}[(\beta u^n_{0x} + \alpha u^{n+1}_1)(\beta u^{p-n}_{1,x} + 2\alpha u^{p-n+1}_2) + (\beta u^n_{1,x} + 2\alpha u^{n+1}_2)(\beta u^{p-n}_{0,x} + \alpha u^{p-n+1}_1)] + 4\alpha u^1_2(\beta u^p_{0x} + \alpha u^{p+1}_1) + 2(\beta u^p_{1,x} + 2\alpha u^{p+1}_2)(\beta u^0_{0x} + \alpha u^1_1) - [cu^p_{0x} + du^p_{1x} + 2bu^{p+1}_2 + au^{p+1}_1] = 0 \tag{2.17}$$

We set

$$g_p = du^p_{1x} - 2(\beta u^p_{1,x} + 2\alpha u^{p+1}_2)(\beta u^0_{0x} + \alpha u^1_1) - \sum_{n=1}^{p-1}[(\beta u^n_{0x} + \alpha u^{n+1}_1)(\beta u^{p-n}_{1,x} + 2\alpha u^{p-n+1}_2) + (\beta u^n_{1,x} + 2\alpha u^{n+1}_2)(\beta u^{p-n}_{0x} + \alpha u^{p-n+1}_1)] + u^p_{1,xx} + 6u^p_3 - \gamma u^p_1 \tag{2.18}$$

hence (2.17) yields

$$4\alpha u^1_2(\beta u^p_{0x} + \alpha u^{p+1}_1) = (cu^p_{0x} + 2bu^{p+1}_2 + au^{p+1}_1) = g_p$$

and by analogy with (2.10) where 2fh is replaced by g_p,

$$u^{p+1}_1 = \frac{u^b_{ox}(c - 4\alpha\beta u^1_2) + g_p + 2bu^{p+1}_2}{\Delta} \tag{2.19}$$

We finally obtain the equation for u^p_0. We write (2.7) for p, and $\ell=0$ and use (2.19). We obtain

$$- u^p_{0xx} + \gamma u^p_0 - 2u^p_2 + \sum_{n=1}^{p-1}(\beta u^n_{0x} + \alpha u^{n+1}_1)(\beta u^{p-n}_{0x} + \alpha u^{p-n+1}_1) + 2(\beta u^p_{0x} + \alpha u^{p+1}_1)(\beta u^0_{0x} + \alpha u^1_1) - (du^p_{0x} + bu^{p+1}_1) = 0$$

But by analogy with (2.13)

$$\beta u_{0x}^{p} + \alpha u_1^{p+1} = \frac{u_{0x}^{p}(\alpha c - \beta a) + \alpha(2bu_2^{p+1} + g_p)}{\Delta}$$

therefore u_0^p as solution of

$$- u_{0xx}^{p} + \gamma u_0^p + u_{0x}^{p}\left[\frac{2(\beta u_{0x}^{0} + \alpha u_1^1)(\alpha c - \beta a)}{\Delta} - d\right] = 2u_2^p + bu_1^{p+1} - 2(\beta u_{0x}^{0} + \alpha u_1^1)\frac{\alpha(2bu_2^{p+1} + g_p)}{\Delta} - \sum_{n=1}^{p-1}(\beta u_{0x}^{n} + \alpha u_1^{n+1})(\beta u_{0x}^{p-n} + \alpha u_1^{p-n+1}) \tag{2.20}$$

3. Study of Function u_ℓ^p

The only problem concerns function u_0^0 which is solution of a non-linear problem. Set $u = u_0^0$ then we can write (2.14) as follows

$$- u'' + \gamma u + u'^2\lambda^2 + \mu u' = \nu \tag{3.1}$$

where $\lambda(x)$, $\mu(x)$, $\nu(x)$ are given functions which are bounded and that we may assume as many times differentiable as we want, and $\gamma > 0$ constant. We can connect to (3.1) a stochastic control problem as follows

$$dy = (\mu(y) + 2\lambda(y)v(t))dt + \sqrt{2}\, dw \qquad y(0) = x \tag{3.2}$$

$$J_x(v(\cdot)) = E\int_0^\infty e^{-\gamma t}[\nu(y) + v(t)^2]dt$$

$$u(x) = \inf_{v(\cdot)} J_x(v(\cdot)). \tag{3.3}$$

3.1. A Priori Estimates

Lemma 3.1: Assume we have a solution u of (3.1) sufficiently smooth, then

$$\|u\|_{L^\infty} \le \frac{\|\nu\|_{L^\infty}}{\gamma} \tag{3.4}$$

Proof: Follows from the maximum principle. ∎

Lemma 3.2: Same assumption as in Lemma 3.1 then

$$\|u'\|_{L^\infty} \le C.$$

<u>Proof</u>: We follow Ladyzhenakaya Uralt'seva [1]. Define

$$\beta_\rho(x) = \exp - \rho(|x|^2 + 1)^{1/2}$$

$$\beta'_\rho = \frac{-\rho\beta_\rho x}{(|x|^2+1)^{1/2}} .$$

Let $k \ge 1$. Set

$$u = \varphi(v), \quad \varphi \text{ function defined later.}$$

We get from (3.1) and

$$u' = \varphi' v' \qquad u'' = \varphi'' v'^2 + \varphi' v'' - (\varphi'' v'^2 + \varphi' v'') + a(x,u,u_x) = 0$$

where

$$a(x,\theta,p) = \lambda^2(x)p^2 + \mu(x)p + \gamma\theta - \nu(x) \tag{3.5}$$

hence

$$- v'' - \frac{\varphi''}{\varphi'} v'^2 + \frac{a}{\varphi'} = 0 \tag{3.6}$$

The function φ will be chosen such that

$$\varphi' > 0. \tag{3.7}$$

We next set

$$w = v'^2 \tag{3.8}$$

$$\eta = (w-k)^+$$

$$A_k = \{x \mid w(x) > k\}.$$

If A_k is of Lebesgue measure 0, then the result is proved. Let us assume that

it is of Lebesgue measure >0. We multiply (3.7) by $(2v'\eta)'\beta_\rho^2$ other the set A_k, hence

$$\int_{A_k}[-v'' - \frac{\varphi''}{\varphi'} v'^2 + \frac{a}{\varphi'}](2v'\eta)'\beta_\rho^2\,dx = 0. \tag{3.9}$$

Since η is 0 on the boundary of A_k, we can integrate by parts

$$-\int_{A_k}\beta_\rho^2[-v'' - \frac{\varphi''}{\varphi'} v'^2 + \frac{a}{\varphi'}]'2v'\eta\,dx + \int_{A_k}\frac{2\rho x\beta_\rho^2}{(1+|x|^2)^{1/2}}[-v'' - \frac{\varphi''}{\varphi'} v'^2 + \frac{a}{\varphi'}]2v'\eta = 0 \tag{3.10}$$

We use

$$\int_{A_k}\beta_\rho^2 v'''2v'\eta\,dx = -\int_{A_k}\beta_\rho^2 v''(2v'\eta)'dx + \int_{A_k}\frac{2\rho x\rho_\rho^2}{(1+|x|^2)^{1/2}} v''(2v'\eta)dx \tag{3.11}$$

hence

$$0 = -\int_{A_k}\beta_\rho^2 v''(2v'\eta)'dx + \int_{A_k}\rho_\rho^2 2v'\eta[(\frac{\varphi''}{\varphi'}v'^2 - \frac{a}{\varphi'})' + \frac{2\rho x}{(1+|x|^2)^{1/2}}(-\frac{\varphi''}{\varphi'}v'^2 + \frac{a}{\varphi'})]dx \tag{3.12}$$

$$0 = \int_{A_k}\rho_\rho^2\{-w'^2 + (w-k)[-2v''^2 + 2(\frac{\varphi''}{\varphi'})'w^2 + 2\frac{\varphi''}{\varphi'}v'w' - 2v'(\frac{a}{\varphi'})' + \frac{2\rho x}{(1+|x|^2)^{1/2}}(-\frac{\varphi''}{\varphi'}w + \frac{a}{\varphi'})]\}dx$$

or using $2v''^2 = \frac{w'^2}{2w}$

$$\int_{A_k}\beta_\rho^2[w'^2 + (w-k)\frac{w'^2}{2w} - 2(w-k)(\frac{\varphi''}{\varphi'})'w^2]dx = \int_{A_k}\beta_\rho^2(w-k)[2\frac{\varphi''}{\varphi'}v'w' - 2\frac{v'}{\varphi'}\frac{da}{dx} + 2aw\frac{\varphi''}{\varphi'^2} + \frac{2\rho x}{(1+|x|^2)^{1/2}}(-\frac{\varphi''}{\varphi'}w + \frac{a}{\varphi'})]dx \tag{3.13}$$

We have

$$a(x) = \lambda^2(x)\varphi'^2 w + \mu\varphi' v' + \gamma u - \nu$$

$$\frac{da}{dx} = 2\lambda\lambda'\varphi'^2 w + \lambda^2 2\varphi'\varphi'' v' w + \lambda^2\varphi'^2 w' + \mu'\varphi' v' + \mu\varphi'' w + \mu\varphi' v'' + \gamma\varphi' v' - \nu'$$

$$\frac{v'}{\varphi'}\frac{da}{dx} = 2\lambda\lambda'\varphi'v'w + 2\lambda^2\varphi''w^2 + \lambda^2\varphi'v'w' + \mu'w + \mu\frac{\varphi''}{\varphi'}v'w + \frac{\mu}{2}w' + \gamma w - \frac{\nu'}{\varphi'}v'$$

In the right hand side of (3.13) we have terms involving w', namely

$$(w-k)[2\frac{\varphi''}{\varphi'} + \lambda^2\varphi']\, v'w' \le C[\delta(w-k)\frac{w'^2}{w} + \frac{1}{\delta}(w-k)w^2(\frac{\varphi''^2}{\varphi'^2} + \varphi'^2)]$$

$$(w-k)\,\frac{\mu}{2}\,w' \le C(w-k)[\delta\frac{w'^2}{w} + \frac{1}{\delta}w]$$

Since $w \ge 1$ on A_k, we majorize the other terms by

$$C(w-k)[\,|\varphi''|w^2] + C_\varphi(w-k)w^{3/2}$$

Going back to (3.13) we obtain choosing δ small

$$\int_{A_k}\beta_\rho^2[w'^2 - 2(w-k)(\frac{\varphi''}{\varphi'})'w^2] \le \int_{A_k}\beta_\rho^2(w-k)[Cw^2(\frac{\varphi''^2}{\varphi'^2} + \varphi'^2 + |\varphi''|) + C_\varphi w^{3/2}]dx \qquad (3.14)$$

The constant C_φ depending on the bounds on φ, φ', φ'', $\frac{1}{\varphi}$, but not constant C.

Let M such that $\|u\| \le M$. We choose

$$\varphi(t) = -2M + 6Me\int_0^t e^{-s^q}ds \quad , \; q \ge 1$$

define t_1, t_2 such that

$$\int_0^{t_1} e^{-s^q}ds = \frac{1}{6e} \qquad \int_0^{t_2} d^{-s^q}ds = \frac{1}{2e}$$

$$\varphi(t_1) = -M \qquad \varphi(t_2) = M$$

$$-M \le \varphi(t) \le M \qquad \text{for} \qquad t_1 \le t \le t_2.$$

We have $t_1 > \frac{1}{6e}$ and $t_2 < \frac{1}{2e}$ since

$$\int_0^{1/2} e^{-s^q}ds > \frac{1}{2e}\ .$$

so for a choice of q to be made later, we have

$$\frac{1}{6e} < t_1(q) < t < t_2(q) < \frac{1}{2e}$$

if $-M < \varphi(t) < M$. Now

$$\varphi' = 6Me\ e^{-t^q} \qquad \varphi'' = -6Meq t^{q-1} e^{-t^q}$$

$$\frac{\varphi''}{\varphi'} = -q t^{q-1} \qquad \left(\frac{\varphi''}{\varphi'}\right)' = -q(q-1)t^{q-2}.$$

We have $\varphi' > 0$ φ, φ', φ'', $\frac{1}{\varphi}$, bounded $-\left(\frac{\varphi''}{\varphi'}\right)' > 0$. Let us now choose q such that

$$-\left(\frac{\varphi''}{\varphi'}\right)' > C\left(\frac{\varphi''^2}{\varphi'^2} + \varphi'^2 + |\varphi''|\right) \tag{3.15}$$

or

$$q(q-1)t^{q-2} > C[q^2 t^{2q-2} + 36M^2 e^2 e^{-2t^q} + 6Meq t^{q-1} e^{-t^q}]$$

which is satisfied for q large enough. Therefore we deduce from (3.14)

$$\int_{A_k} \beta_\rho^2 [w'^2 - (w-k)\left(\frac{\varphi''}{\varphi'}\right)' w^2]dx \le \int_{A_k} \beta_\rho^2 (w-k) C_\varphi w^{3/2} dx \tag{3.16}$$

hence since $w \ge k$ and $-\left(\frac{\varphi''}{\varphi'}\right)' \ge C'_\varphi > 0$

$$k^{1/2} \le \frac{C_\varphi}{C'_\varphi}.$$

Therefore if $k^{1/2} > \frac{C_\varphi}{C'_\varphi}$, the set A_k is of measure 0. This proves that $\frac{C_\varphi}{C'_\varphi}$ is a bound for w. ∎

3.2. Existence and Uniqueness

<u>Theorem 3.1</u>: <u>Assume that the function</u> λ, μ, ν <u>in</u> (3.1) <u>are</u> C^1, <u>bounded with bounded derivatives</u>. <u>Then there exists one and only one solution of</u> (3.1) <u>which is</u> C^3 <u>bounded as well at its derivatives</u>.

<u>Proof</u>: Let $\theta(z)$ be a smooth function such that

$$\theta(z) = z \qquad \text{if} \qquad (z) \leq k$$

$$|\theta'| \leq 1$$

$$\theta \text{ bounded } |\theta(z)| \leq \mathrm{Min}(|z|, C)$$

We consider the equation

$$-u'' + \gamma u + \theta(u')^2\lambda^2 + \mu u' = \nu \tag{3.17}$$

The nonlinear term

$$H(x,p) = \theta(p)^2\lambda^2(x)$$

is Lipschitz in p, since

$$H_p = 2\theta\, \theta_p \lambda^2(x).$$

Therefore there is existence and uniqueness of the solution of (3.17) (although since $\theta(p)^2$ is not convex equation (3.17) does not correspond a priori to a control problem).

The solution of (3.17) is C_b^3.[1] Now redoing the calculation of Lemma 3.2, by virtue of the assumptions of θ, once easily checks that the same estimates remain valid. Now if k is the bound on $|u'|$ obtained in Lemma 3.2, we see that $\theta(u') = u'$. Hence the existence. Uniqueness is the consequence of the maximum principle. Indeed if u, $\tilde{u}$ are two solutions then setting

$$v = u - \tilde{u}$$

we have by difference

---

[1] C_b^n = space of functions with n derivatives continuous and bounded.

$$-v'' + \gamma v + \lambda^2\ v'(u' + \tilde{u}') + \mu v' = 0$$

and since u', $\tilde{u}'$ are bounded the maximum principle shows that $v = 0$. ∎

Theorem 3.2. Under the assumptions of Theorem 3.1, (3.3) holds and there exists an optimal control.

Proof: More precisely, we prove that (3.3) holds for the following class of admissible controls: $v(t)$ is non anticipative.

Then the solution of (3.2) is defined in the space $\mathcal{H} = \{y \mid E \sup_{0 \le t \le T} |y(t)|^2 < \infty\}$, $\forall T$ finite.

By virtue of the regularity properties of the solution of (3.1), the standard theory of Stochastic Control (see Fleming-Rishel [4]) yields the desired result. The optimal control is defined by the following feedback rule

$$\hat{u}(t) = -\lambda(\hat{y}(t))u'(\hat{y}(t)) \tag{3.18}$$

where $\hat{y}(t)$ is the optimal state. ∎

Turning back to function u^ρ_ℓ we have

Theorem 3.4. Assuming all functions of x entering in (1.2), (1.3) C^1_b, then u^0_0 is uniquely defined by (2.14) in C^3_b, and u^1_2, u^1_1 are uniquely defined and C^1_b. If the coefficients are C^2_b then u^2_3, u^2_2, u^2_1, u^1_0 are well defined and C^0_b. If the coefficients are sufficiently smooth, we can define in a unique way the functions u^b_ℓ up to a given index p.

Proof: Assume the coefficients to be C^2_b (for instance), u^0_0 is then in C^4_b, and $u^1_2 \in C^2_b$, $u^1_1 \in C^2_b$. From (2.15) with $r = 1$, one obtains $u^2_3 \in C^1_b$, from (2.16) with $r = 1$, $\ell = 2$ one obtains $u^2_2 \in C^0_b$, from (2.19) one obtains $u^2_1 \in C^0_b$ and from (2.20) $u^1_0 \in C^2_b$. Clearly we can make an induction argument. ∎

4. Interpretation of the Limit Problem

We now give the interpretation of $u_o^o = u$ with respect to the original control problem (1.2), (1.3). This is the reduced control problem. The reduced control problem consists in setting $\epsilon = 0$, after multiplication by ϵ, i.e.,

$$ax + b + 2\alpha v = 0$$

hence

$$z = -\frac{b+2\alpha v}{a}$$

and using this in the slow system equation, we obtain

$$dx = \left(-\frac{c}{a}(b+2\alpha v) + d + 2\beta v\right)dt + \sqrt{2}\,dw \tag{4.1}$$

$$x(0) = x$$

$$J_x^o(v(\cdot)) = E\int_o^\infty e^{-\gamma t}\left[\left(f - \frac{h}{a}(b+2\alpha v)\right)^2 + v^2\right]dt.$$

To see the connection with (2.14) and (3.2) under a suitable choice of λ, μ, ν we make the following change of control variable

$$v = \tilde{v}\,\frac{a}{\Delta} + 2\,\frac{(fa-bh)\alpha h}{\Delta^2} \tag{4.2}$$

then after easy calculations, one can rewrite problem (4.1) as follows

$$dx = \left(\frac{da-bc}{a} + \frac{4}{a}\,\frac{(\beta a - \alpha c)}{\Delta^2}(fa-bh)\alpha h + 2\,\frac{\tilde{v}}{\Delta}\,(\beta a - \alpha c)\right)dt + \sqrt{2}\,dw$$

or as it is easily verified

$$dx = \frac{a(da-bc) + 4h\alpha(f(\beta a - \alpha c) + h(d\alpha - \beta b))}{a^2 + 4\alpha^2 h^2}\,dt + 2\,\frac{(\beta a - \alpha c)}{\sqrt{a^2+4\alpha^2 h^2}}\,\tilde{v} + \sqrt{2}\,dw \tag{4.3}$$

with a payoff functional

$$J_x(\tilde{v}(\cdot)) = E\int_0^\infty e^{-\gamma t}[\tilde{v}(t)^2 + \frac{(fa-bh)^2}{a^2+4\alpha^2h^2}]dt \tag{4.4}$$

This is exactly the form (3.2), (3.3) with the choices

$$\begin{aligned} v &= \frac{(fa-bh)}{a^2+4\alpha^2h^2} \\ \mu &= \frac{a(da-bc)+4h\alpha(f(\beta a-\alpha c)+h(d\alpha-\beta b))}{a^2+4\alpha^2h^2} \\ \lambda &= \frac{\beta a-\alpha c}{\sqrt{a^2+4\alpha^2h^2}} \end{aligned} \tag{4.5}$$

Therefore we may state.

<u>Lemma 4.1</u>: <u>The function</u> $u = u_0^0$ <u>is the Bellman function of a stochastic control problem, which is obtained from</u> (1.2) as follows

$$\begin{aligned} dx &= (cz+d+2\beta v)dt + \sqrt{2}\,dw_1 \\ 0 &= az+b+2\alpha v \\ x(0) &= x \end{aligned} \tag{4.6}$$

$$J_x^0(v(\cdot)) = E\int_0^\infty e^{-\gamma t}[(f+hz)^2+v^2]dt. \tag{4.7}$$

<u>The optimal control of this reduced problem is obtained by the following feedback</u>

$$v_s(x) = \frac{(\alpha c-\beta a)}{a^2+4\alpha^2h^2}au_{0x}^0 + 2\,\frac{(fa-bh)\alpha h}{a^2+4\alpha^2h^2} \tag{4.8}$$ [1] ∎

We define $z_s(x)$ as the value of z defined by (4.6) when we apply the feedback control (4.8) $v_s(x)$, namely

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[1] s stands for slow.

$$z_s(x) = -\frac{b+2\alpha v_s}{a} = -\frac{b}{a} - \frac{2\alpha}{a}\left[\frac{(\alpha c - \beta a)}{\Delta^2}\, a u^o_{ox} + 2\,\frac{(fa-bh)\alpha h}{\Delta^2}\right] = \frac{2\alpha}{\Delta^2}(\beta a - \alpha c)u^o_{ox}$$

$$- \frac{b}{a} - \frac{4\alpha^2}{a\Delta^2}(fa - bh)h$$

i.e.,

$$z_s(x) = \frac{2\alpha}{a^2+4\alpha^2h^2}(\beta a - \alpha c)u^o_{ox} - \frac{4\alpha^2 fh}{\Delta^2} - \frac{ab}{\Delta^2} \tag{4.9}$$

We now introduce the composite control as in Chow-Kokotovic [2], [3]. Going back to the optimal feedback (1.8), we consider the first term of its expansion obtained from (2.4), namely

$$v_c(x,z) = -(\beta u^o_{ox} + \alpha u^1_1) - 2\alpha z u^1_2 \tag{4.10}$$

and using (2.13) we obtain

$$v_c(x,z) = \frac{u^o_{ox}(\beta a - \alpha c) - \alpha(2bu^1_2 + 2fh)}{\Delta} - 2\alpha z u^1_2 \tag{4.11}$$

Then we have

Lemma 4.2: The composite control $v_c(x,z)$ can be written as follows

$$v_c(x,z) = v_s(x) - 2\alpha u^1_2(z - z_s). \tag{4.12}$$

Proof: We compute from (4.11) and (4.8)

$$v_c(x,z) - v_s(x) = u^o_{ox}(\beta a - \alpha c)\left(\frac{1}{\Delta} + \frac{a}{\Delta^2}\right) - 2\alpha u^1_2 z - \frac{\alpha}{\Delta}(2bu^1_2 + 2fh) + 2\,\frac{\alpha h(-fa+bh)}{\Delta^2}$$

$$= -2\alpha u^1_2 z + \frac{4\alpha^2 u^1_2(\beta a - \alpha c)u^o_{ox}}{\Delta^2} - \frac{2\alpha fh\; 4\alpha^2 u^1_2}{\Delta^2} + 2\alpha b\left(-\frac{u^1_2}{\Delta} + \frac{h^2}{\Delta^2}\right) \tag{4.13}$$

But

$$2\alpha b(-\frac{u_2^1}{\Delta}+\frac{h^2}{\Delta^2}) = 2\alpha b(-\frac{a+\Delta}{4\alpha^2\Delta}+\frac{h^2}{\Delta^2}) = 2\alpha b(-\frac{a}{4\alpha^2\Delta}-\frac{1}{4\alpha^2}+\frac{h^2}{\Delta^2}) = 2\alpha b(-\frac{a}{4\alpha^2\Delta}$$

$$-\frac{\Delta^2-4\alpha^2h^2}{4\alpha^2\Delta^2}) = 2\alpha b(-\frac{a}{4\alpha^2\Delta}-\frac{a^2}{4\alpha^2\Delta^2}) = -\frac{2\alpha ba}{4\alpha^2\Delta}(1+\frac{a}{\Delta}) = -\frac{2\alpha bau_2^1}{\Delta^2}$$

Taking into account (4.9) we obtain (4.12). ∎

As in Chow-Kokotovic one can interpret the 2nd term in (4.12) as the optimal control for the following control problem.

$$\text{Min} \int_0^\infty (h^2 z_f^2 + v_f^2)\,dt$$

$$\frac{dz_f}{dt} = az_f + 2\alpha v_f$$

$$z_f(0) = 0$$

provided that we interpret $z - z_s$ as z_f (x being frozen).

5. Stabilization Property

Let us prove now that the composite control maintains the initial payoff bounded as $\epsilon \to 0$, provided the coefficients are sufficiently smooth and γ is large enough, but fixed.

Theorem 5.1: Assume the data sufficiently smooth and bounded and γ sufficiently large but fixed. Choose in (1.2) the control to be defined by feedback (4.12). Then the functional (1.3) remains bounded as $\epsilon \to 0$.

Proof: Since the coefficients are smooth enough, we may assume that the functions $v_s(x)$, $z_s(x)$ are sufficiently smooth with bounded derivatives. Using feedback (4.12), we consider the system

$$dx = cz + d + 2\beta v_s - 4\alpha\beta u_2^1(z - z_s) + \sqrt{2}\, dw_1$$

$$dz = \frac{1}{\epsilon}\,[az + b + 2\alpha(v_s - 2\alpha u_2^1(z - z_s))]dt + \sqrt{2}\, dw_2 \qquad (5.1)$$

$$x(0) = 0, \quad z(0) = z.$$

Setting $z - z_s = z_f$, we obtain a pair of stochastic processes $x(t)$, $z_f(t)$ solutions of

$$dx = (cz_s + d + 2\beta v_s + (c - 4\alpha\beta u_2^1)z_f)dt + \sqrt{2}\, dw_1$$

$$dz_f = -\frac{1}{\epsilon}\,\Delta\, z_f dt - z_{sx}dx - z_{sxx}dt + \sqrt{2}\, dw_2.$$

In deriving the 2nd equation we have used the fact that

$$az_s + b + 2\alpha v_s = 0$$

and

$$a - 4\alpha^2 u_2^1 = -\Delta$$

and by Ito's formula

$$dz_s = +z_{sx}dx + z_{sxx}dt$$

where z_{sx}, z_{sxx} stand for the derivatives of z_s with argument $x(t)$.

We thus have simplifying notation

$$dx = (\ell(x) + \lambda(x)z_f)dt + \sqrt{2}\, dw_1$$

$$dz_f = -\frac{1}{\epsilon}\,\Delta z_f dt + (m(x) + \delta(x)z_f)dt$$

$$-\sqrt{2}\, z_{sx}\, dw_1 + \sqrt{2}\, dw_2$$

where $\ell(x)$, $\Upsilon(x)$, $m(x)$, $\delta(x)$ are bounded functions.

From Ito's formula we have

$$e^{-\gamma t}(|x(t)|^2+|z_f(t)|^2) = |x|^2+|z-z_s(x)|^2+\int_0^t e^{-\gamma s}[-\gamma(|x(s)|^2+|z_f(s)|^2)$$

$$+(2+z_{sx}{}^2)ds]+\int_0^t e^{-\gamma s}(2x(s)dx(s)+2z_f(s)dz_f(s))$$

Set $\varphi(t) = |x(t)|^2+|z_f(t)|^2$, we obtain by taking expectations

$$e^{-\gamma t}E\varphi(t)+\gamma E\int_0^t e^{-\gamma s}\varphi(s)ds+\frac{2}{\epsilon}E\int_0^t \Delta z_f^2(s)ds \le C\,E\int_0^t e^{-\gamma s}\varphi(s)ds+C+|x|^2+|z|^2$$

where C is a constant depending only on the bounds of m, δ, ℓ, λ and z_s. Therefore if γ is large enough but fixed, it follows that

$$E\int_0^\infty e^{-\gamma t}(|x(t)|^2+|z_f(t)|^2)dt < C \text{ independent of } \epsilon,$$

from which it follows that

$$E\int_0^\infty e^{-\gamma t}(|x(t)|^2+|z(t)|^2)dt < \infty.$$

Therefore

$$\inf_{v(\cdot)} J^\epsilon_{x,z}(v(\cdot)) \le C \quad \text{independent of } \epsilon. \tag{5.2}$$

The constant of course depends on the initial condition x,z. In fact we have

$$0 \le \inf_{v(\cdot)} J^\epsilon_{x,z}(v(\cdot)) \le C+|x|^2+|z|^2 \tag{5.3}$$

■

<u>Lemma 5.1</u>: <u>When we apply the composite control, the corresponding states</u> x, z <u>satisfy</u>

$$E\int_0^\infty e^{-\gamma t}(|x(t)|^4+|z(t)|^4)dt \le C \quad (\text{independent of } \epsilon). \tag{5.4}$$

Proof: Similar to that of Theorem 5.1, we apply Ito's formula to $e^{-\gamma t}(|x(t)|^4 + |z(t)|^4)$ and use the estimate already obtained

$$E\int_0^\infty e^{-\gamma t}(|x(t)|^2 + |z(t)|^2)dt < \infty. \quad \blacksquare$$

Let us now introduce

$$w^\epsilon(x,z) = \inf_{v(\cdot)\in V} J^\epsilon_{x,z}(v(\cdot)) \tag{5.5}$$

where

$$V = \{v(t) \mid E\int_0^\infty e^{-\gamma t}|v(t)|^2 dt < M,\ E\int_0^\infty e^{-\gamma t}(\ x(t)|^4 + |z(t)|^4)dt \leq C\} \tag{5.6}$$

The constants being chosen such that the control $v^c(t)$ obtained from the composite feedback (4.12) belongs to (5.6) for any ϵ. This is possible by virtue of Theorem 5.1 and Lemma 5.1. Then we can state

<u>Theorem 5.2</u>: <u>We make the assumptions of Theorem</u> 5.1, <u>then</u>

$$w^\epsilon(x,z) \to u_o^o(x) \qquad \text{pointwise as } \epsilon \to 0.$$

<u>Proof</u>: Consider

$$\varphi^\epsilon(x,z) = u_o^o(x) + \epsilon(u_0^1(x) + zu_1^1(x) + z^2u_2^1(x)).$$

Let $v(t)$ be a control belonging to V and let $x(t)$, $z(t)$ be the corresponding solution of (1.2) (depending of course on ϵ). By Ito's formula

$$E\varphi^\epsilon(x(t),z(t))e^{-\gamma t} = \varphi^\epsilon(x,z) + E\int_0^t e^{-\gamma s}[-\gamma\varphi^\epsilon(x(s),\ z(s))$$

$$+ \frac{\partial\varphi^\epsilon}{\partial x}\cdot(cz + d + 2\beta v) + \frac{\partial\varphi^\epsilon}{\partial z}\cdot\frac{1}{\epsilon}(az + b + 2\alpha v) + \Delta\varphi^\epsilon(x(s),\ z(s))]ds.$$

We can let $t \to +\infty$, and deduce

$$\varphi^{\epsilon}(x,z) = E\int_0^{\infty} e^{-\gamma t}[\gamma u_o^o + \epsilon(u_0^1 + zu_1^1 + z^2u_2^1) - (cz+d+2\beta v)(u_{ox}^o + \epsilon(u_{0x}^1 + zu_{1x}^1$$

$$+ z^2u_{2x}^1)) - (az+b+2\alpha v)(u_1^1 + 2zu_2^1) - u_{0xx}^0 - \epsilon(u_{0xx}^1 + zu_{1xx}^1 + z^2u_{2xx}^1) - 2\epsilon z u_2^1]dt \tag{5.7}$$

and using the equations for u_0^0, u_1^1, u_2^1, we can write (5.7) as

$$\varphi^{\epsilon}(x,z) = E\int_0^{\infty} e^{-\gamma t}[(f+hz)^2 - (\beta u_{0x}^0 + \alpha u_1^1 + 2\alpha z u_2^1 + v)^2 + v^2 + \epsilon(u_0^1 + zu_1^1 + z^2u_2^1)$$

$$- \epsilon(cz+d)(u_{0x}^1 + zu_{1x}^1 + z^2u_{2x}^1) - \epsilon(u_{0xx}^1 + zu_{1xx}^1 + z^2u_{2xx}^1) - 2\epsilon z u_2^1]dt \tag{5.8}$$

and since the control belongs to V,

$$\varphi^{\epsilon}(x,z) \le J_{x,z}^{\epsilon}(v(\cdot)) + c\,\epsilon$$

therefore letting $\epsilon \to 0$

$$u_0^0(x) \le \underline{\lim}\; w^{\epsilon}(x,z) \tag{5.9}$$

we now consider (5.8) with v equal to the composite control. We deduce from (5.8)

$$\varphi^{\epsilon}(x,z) = J_{x,z}^{\epsilon}(v^c) + 0(\epsilon) \ge w^{\epsilon}(x,z) + 0(\epsilon)$$

hence

$$u_0^0(x) \ge \overline{\lim}\; w^{\epsilon}(x,z)$$

from which and (5.9) we obtain the derived result. ∎

Conclusion

In the deterministic problem considered by Chow and Kokotovic, there was no discount, and of course no noise. They had to assume that the reduced control problem has a solution. Using the composite control they derive

under suitable assumptions results similar to those of Theorem 5.1 and 5.2 (stabilization results). We have shown here that in the stochastic case (with perfect information), the discounted problem is well posed. This allows us to prove stabilization results, with the only assumptions that there is sufficient smoothness on the data, and that the discount factor is sufficiently large (but fixed). Our proof is completely different from that of Chow-Kokotovic.

Extensions of this work can be done in the following directions. We can use a different scaling between noises and the dynamics of the system. It seems reasonable to take $\frac{\sqrt{2}\, dw_2}{\sqrt{\epsilon}}$ instead of $\sqrt{2}\, dw_2$ in the fast system equation. One can consider the vector case, i.e. x, z are vectors as in Chow-Kokotovic. It would be nice to remove the assumption that γ is sufficiently large.

It would be also interesting to study the Bellman equation for the ϵ problem.

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